Order Types
(from geometry to combinatorics)

Triangulations

• Triangulation of a point set \( S \):
  – Maximal plane straight-line graph with vertex set \( S \)
  – Decomposition of the convex hull of \( S \) into non-overlapping triangles

Triangulations

• Delaunay Triangulation
  – Dual to Voronoi diagram
• Greedy Triangulation
• Minimum Weight Triangulation
  – One of the 14 open problems in Garey and Johnson's book on NP-Completeness
  – Solved: Minimum Weight Triangulation is NP-hard
    [W. Mulzer, G. Rote, SoCG June 2006]

• How many triangulations do exist?

How Many Triangulations?

\[ B(m) = \sum_{k=1}^{m-1} B(k) \cdot B(m-k) ; m\geq n-1 \]

\[ T(n) = B(m) + \ldots + C_{m-1} = C_{n-2} \]

Catalan numbers:

\[ C_0 = 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots \]

• Belong to the 'top five' numbers in Combinatorics:
  • Complete Binary trees with \( n \) leaves \( C_n \)
  • Completely parenthesizing a product of \( n \) letters \( C_n \)
  • Two (robot) soccer teams play \( n: n \). How many goal sequences exist, such that Team A is never behind Team B? \( \Rightarrow C_n \)
  • Triangulations \( C_{n-2} \)

How Many Triangulations?

Number of triangulations of a convex \( n \)-gon

Number of complete binary trees with fixed root and \( n-1 \) leaves

\[ \Rightarrow \]
“And of course there are much more triangulations in the general case!”

Of Course There Are More …

n = 3

n = 4

Well, not really. Maybe asymptotically …

n = 5

Stop: n=5?

How Many Point Sets?

How many ways to draw \( n \) points in the plane?

What is the probability, that any four points chosen at random (independently, uniformly) from a planar region are in convex position?

Why Crossing Properties?

Stop: Count! \( n = 3, 4, 5 \)

Crossing Properties

- point sets in the real plane \( \mathbb{R}^2 \)
- finite point sets of fixed size
- point sets in general position
- point sets with different crossing properties
Crossing Properties

- Line segments $ab$, $cd$ crossing $\iff$ different orientations $abc, abd$ and different orientations $cda, cdb$

Order Type

- Order type of point set: mapping that assigns to each ordered triple of points its orientation [Goodman, Pollack, 1983]

  - Orientation:
    - Left/positive
    - Right/negative

- Not really equivalent: provide an order type of size 4, which cannot be a point set!

- $n=3$: 1 order type
- $n=4$: 2 order types

How to decide whether two point sets are of the same order type?

- Encoding order types: $\lambda$-matrix
  
  - $S = \{p_1, \ldots, p_n\}$.. labelled point set
  
  - $\lambda(i,j)$ .. number of points of $S$ on the left of the oriented line through $p_i$ and $p_j$

- Theorem: order type $\iff \lambda$-matrix

Stop: Why true?
Order Type

- natural $\lambda$-matrix: $p_1$ on the convex hull, $p_2, \ldots, p_n$ sorted clockwise around $p_1$
- lexicographically minimal $\lambda$-matrix: unique "fingerprint" for an order type
- same order type $\iff$ identical lexicographically minimal $\lambda$-matrices

Properties of the natural $\lambda$-matrix:
- $\lambda(i,j)=0 \Rightarrow p_i$ and $p_j$ on the convex hull
- $\lambda(i,j)+\lambda(j,i) = n - 2$
- $p_1$ on the convex hull
- $p_2, \ldots, p_n$ sorted clockwise around $p_1$
- lexicographically minimal matrix

Enumerating Order Types

- Geometrical insertion:
  - for each order type of $n$ points consider the underlying line arrangement
  - insert a point in each cell of each line arrangement $\Rightarrow$ order types of $n+1$ points

- Generating order types, iterative approach:
  - complete order type extension:
    - input: order type $S_n$ of $n$ points
    - output: all different order types $S_{n+1}$ of $n+1$ points that contain $S_n$ as a sub-order type
Enumerating Order Types

5 points: 3 order types

Is geometrical iterative insertion complete?

Line arrangement not unique for fixed order type!

point-line duality: \( p \leftrightarrow T(p) \)

\[ T: p \rightarrow p' l = 1, l = (x, y) \]

Properties:

\( bc \leftrightarrow ac \leftrightarrow ab \)
Enumerating Order Types

point-line duality: \( p \leftrightarrow T(p) \)

order type \( \leftrightarrow \) local intersection sequence
(point set) \( \leftrightarrow \) (line arrangement)

Enumerating Order Types

point-line duality: \( p \leftrightarrow T(p) \)

order type \( \leftrightarrow \) local intersection sequence
(point set) \( \leftrightarrow \) (line arrangement)

Enumerating Order Types

again: (line) arrangement

pseudoline arrangement

Enumerating Order Types

point-line duality: \( p \leftrightarrow T(p) \)

order type \( \leftrightarrow \) local intersection sequence
(point set) \( \leftrightarrow \) (line arrangement)

abstract \( \leftrightarrow \) local intersection sequence
order type \( \leftrightarrow \) (pseudoline arrangement)

Enumerating Order Types

abstract duality for natural \( \lambda \)-matrices

Find the error!
Enumerating Order Types

Abstract order type extension algorithm:
- duality abstract order type ↔ pseudoline arrangement
- extend pseudoline arrangement with an additional pseudoline in all combinatorial different ways (local intersection sequences)

Realizability

- Realizability of abstract order types ⇔ stretchability of pseudoline arrangements
- Deciding stretchability is NP-hard. [Mnev, 1985]
- Every arrangement of at most 8 pseudolines in \( \mathbb{R}^2 \) is stretchable. [Goodman, Pollack, 1980]
- Every simple arrangement of at most 9 pseudolines in \( \mathbb{R}^2 \) is stretchable except the simple non-Pappus arrangement. [Richter, 1988]
Realizability

- Group abstract order types into projective classes
  - combinatorial simulation of the rotation (of $\lambda$-matrices) without (knowledge of) realization
  - either all or none elements of a projective class are realizable
- heuristics:
  - geometrical insertion + local optimization (simulated annealing)
- heuristics for proving non-realizability:
  - linear system of inequations derived from Grassmann-Plücker equations [Bokowski, Richter, 1990]

Order Type Data Base

Data Base:

Complete and reliable data base of all different order types of size up to 11 in nice 16-bit integer coordinate representation

(Using natural labeling and lexicographical minimal lambda matrices)

Applications

Motivation for using the data base:

- find counterexamples
- computational proofs
- new conjectures
- structural insight
Crossing Families

A crossing family is a set of $k$ pairwise intersecting line segments. What is the minimum number $n(k)$ of points such that any point set of size at least $n(k)$ admits a crossing family of size $k$?

Crossing family for $k = 2$:

Think!

$k = 3$:

- Any set of $n \geq 10$ points contains at least 3 pair wise crossing edges.
- The bound on $n$ is tight.
- Best previous bound on was $n \geq 37$ [Tóth, Valtr, 1998]
- New 2015: $k = 4$: $n \geq 15$ points contain at least 4 pair wise crossing edges

Why interesting? ⇒ Proofs using divide & conquer

Lower bound for triangulations

Ramsey Type Results

Further Ramsey-type results observed from the database:

⇒ Every set of 8 points contains either an empty convex pentagon or two disjoint empty convex quadrilaterals

Provided 'human readable' proof afterwards. [Aichholzer, Huemer, Kappes, Speckmann, Tóth, 2005/06]

Decomposition

Convex decomposition

Pseudo-Convex decomposition

Further Ramsey-type results observed from the database:

⇒ Every set of 11 points contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral

Provided 'human readable' proof afterwards. [A, Huemer, Kappes, Speckmann, Tóth, 2005/06]
Constant ⇒ Asymptotics

Ramsey-type results for constant size objects might lead to improvements for arbitrary size.

We ‘win’ 3 faces per 9 points …

Decomposition

Convex decomposition Pseudo-Convex decomposition

Triangulations: $D_T = 2n - h - 2$
Pseudo-Triangulations: $D_{PT} = n - 2$

Abstract Extension

- How to generate order types for $n \geq 12$:
  - Complete extension intractable
    - $\approx 750$ billion order types for $n = 12$
    - Disk space > 30 TB
    - Realizability problem
    - $> 200$ years of computation
  - Partial Extension on “suitable” sets

Abstract Extension

- Sub-Set property: For a set of $n$ points with a required property there exists at least one set of $n-1$ points with an according property.
  - Choose unique predecessor for each set (paternity test), proceed tree-like (generate each set only once!)
  - Use distributed computing: each set of the starting base can be extended independently

Happy End Problem

- “Happy End Problem”:
  What is the minimum number $g(k)$ s.t. each point set with at least $g(k)$ points contains a convex $k$-gon?

  - No exact values $g(k)$ are known for $k \geq 7$.
  - Conjecture: $g(k) = 2^{k-2} + 1$ (g(6)=17 solved 2006)

  [Erdős, Szekeres, A combinatorial problem in geometry. 1935]

Happy End Problem

- Order type extension (6-gon problem):
  - Enumerate all order types that do not contain a convex 6-gon
  - Subset property: $S_n$ contains no convex 6-gon $\Rightarrow$ each subset $S_{n+1}$ contains no convex 6-gon

  - Start:
    - $n = 11$ ... 235 987 328 order types
    - $n = 12$ ... 14 048 972 314 (abstract) order types
    - $n = 13$ ... $\approx 800$ 109 (abstract) order types

  - Use improved algorithms and heavy distributed computing
**Crossing Number**

\[ cr(K_n) \] ... rectilinear crossing number of \( K_n \)

minimum number of edge crossings attained by a drawing of \( K_n \) with straight lines

Stop: \( n = 6? \)

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**Crossing Number: History**

- Turán's brick factory problem, \( cr(K_{m,n}) \), 1944
  - Bring bricks on small vehicles from ovens to stores, each oven was connected with each store by rail. Troubles occurred at crossings: cars jumped out, bricks fell down

- Subset property:
  - Each \( n - 1 \) point subgraph of \( K_n \) contains at least \( cr(K_{n-1}) \) crossings, each being counted \( n - 4 \) times:
  - A drawing of \( K_n \) must contain a drawing of \( K_{n-1} \) with \( d_n \) drawings.
  - Parity property for \( n \) odd:
    \[
    cr(K_n) \equiv \begin{cases} 
    0 & \text{if } n \equiv 1 \pmod{4} \\
    1 & \text{if } n \equiv 3 \pmod{4} 
    \end{cases}
    \]

- [Erdős, Guy, 1960-1973]
  - [Brodsky, Durocher, Gether, 2001]
  - [AAK, 2001]
  - [AAK, 2004]

- Drawing with 153 crossings
  \[ cr(K_{12}) = 153 \]

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**Crossing Number: History**

- [Erdős, Guy, 1960-1973]
  - [Brodsky, Durocher, Gether, 2001]
  - [AAK, 2001]
  - [AAK, 2004]

- Table:

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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<th>11</th>
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<td>3</td>
<td>9</td>
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<td>36</td>
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<tr>
<td>( d_n )</td>
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<td>1</td>
<td>3</td>
<td>2</td>
<td>10</td>
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<td>374</td>
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</tbody>
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**Crossing Number**

- Subset property:
Crossing Number

- \( \text{cr}(K_{13}) = 229 \) ?
  - \( K_{13} \) .. 227 crossings \( \Rightarrow \) \( K_{12} \) .. \( \leq 157 \) crossings
  - \( K_{12} \) .. 157 crossings \( \Rightarrow \) \( K_{11} \) .. \( \leq 104 \) crossings

- \( d_{13} \) = ?
  - \( K_{13} \) .. 229 crossings \( \Rightarrow \) \( K_{12} \) .. \( \leq 158 \) crossings
  - \( K_{12} \) .. 158 crossings \( \Rightarrow \) \( K_{11} \) .. \( \leq 104 \) crossings

<table>
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</table>

Crossing Number

Extension of the complete data base:
2 334 512 907 order types for \( n = 11 \)

Extension for rectilinear crossing number:

- \( \text{cr}(K_{13}) = 1029 \)
  - \( \text{cr}(18) = 1029 \); RCN Dec. 2006 (new 2014/15: unique)
  - \( \text{cr}(19), \text{cr}(21) \) using triangular structure [AGOR, 2006]
  - \( \text{cr}(20), \text{cr}(22), \text{cr}(27) \) and \( \text{cr}(30) \) using best sets and pseudo lines / circular sequences [AMLS, 04/2007]
  - \( \text{cr}(28) \in \{7233, 7234\}, \text{cr}(29) \in \{8421, 8423\} \)

Meanwhile obtained results:

- Conjecture: For any \( n \) there always exists an optimal set which contains an optimal \( (n-1) \)-set:
  - \( n = 18 \) provides a counterexample

Structural results:

- \( \text{cr}(K_{18}) = 1029 \) (New: unique optimal example)
Crossing Number

rectilinear crossing constant:

\[ \bar{v}(n) = \frac{cr(K_n)}{4} \]

\[ \bar{v}^* = \lim_{n \to \infty} \bar{v}(n) \]

Lower Bounds on \( \bar{v}^* \)

- \( > 0.3001 \) [Brodsky, Durocher, Gethner, 2001]
- \( > 0.3115 \) [Ai, Aurenhammer, Krasser, 2002]
- \( > 0.3286 \) [Wagner, 2003]
- \( > 0.3328 \) [Ai, Aurenhammer, Krasser, 2003]
- \( \geq 0.3750 \) [Ábrego,Fernández-Merchant, 2003]
- \( > 0.3753 \) [Balogh, Salazar, 2004]
- \( > 0.37968 \) [Ai, García, Orden, Ramos, 2006][A,B,F-M,L,S]
- \( > 0.379972 \) [Ábrego,Fernández-Merchant, Leaños, Salazar, 2008]

Upper Bounds on \( \bar{v}^* \)

- \( < 0.38888 \) [Jensen, 1974]
- \( < 0.38460 \) [Singer, 1971, Manuscript]
- \( < 0.38380 \) [Brodsky, Durocher, Gethner, 2001]
- \( < 0.38074 \) \( n=36, cr=21191 \) [A, Aurenhammer Krasser, 2002]
- \( < 0.38058 \) \( n=54, cr=115999 \) [A, Krasser,2005]
- \( < 0.38055 \) \( n=30, cr=9726 \) [Ábrego,Fernández-Merchant, 2006]
- \( < 0.38054 \) \( n=90, cr=951534 \) [A, Kornberger 2006/07]
- \( < 0.380473 \) \( n=75, cr=450 492 \) [R.Fabila-Monroy, J.Lopez, 2014]
- \( < 0.3804499.. \) \( n=651, cr=2 812 785 506 \) [ongoing joint project]

How Many Point Sets?

What is the probability, that any four points chosen at random (independently, uniformly) from a planar region are in convex position?

[Sylvester, 1865]

- choose independently uniformly at random from a set \( R \) of finite area, \( q^* = \inf q(R) \)
- Convex sets: \( 2/3 \leq q(R) \leq 1-35/(12\pi^2) = 0.7 \)

Sylvester’s Four Point Problem

- Infimum for open sets: \( q^* = \bar{v}^* \) [Scheinerman,Wilf,1994]

Crossing Number

- How to construct asymptotically good drawings:
  - take 'best' known small drawing \( D \)
  - recursively replace each vertex of \( D \) by a copy of \( D \)
- [Singer 1971]
Halving property:
the replacement
halves the remaining
set into groups of
sizes \( \frac{n}{2} \) and \( \frac{n}{2}-1 \)
\((n \text{ even})\)

- Are crossing-optimal abstract order types always realizable?
  - For \( n \leq 18 \): yes. Open for general \( n \geq 19 \) (Example??)
- Do crossing-optimal drawings always contain optimal sub-drawings?
  - For \( n \leq 17 \): yes. No for \( n = 18 \)
    - \( n = 18 \): unique optimal example has NO optimal \( n=17 \) sub-set
- Do optimal drawings have 3 extreme vertices?
  - Yes! [A. García, Orden, Ramos, 2006]
- True constant in asymptotics?
  - Best bounds: 0.379972 ... 0.38045 (gap reduced from 1/10 to 1/2000 or 0.13%)
- Value of \( cr(K_n) \) for larger \( n \) ...

The database is used for ...

- rectilinear crossing number
- crossing families
- Happy End Problem
- convex \( k \)-gons
- empty convex \( k \)-gons
- convex covering
- convex partitioning
- convex decomposition
- number of triangulations
- flip distances
- isomorphic triangulations
- sequential triangulations
- minimum pseudo-triangulations
- Hamiltonian cycles
- min. reflex polygonalizations
- spanning trees
- crossing-free matchings
- \( k \)-sets
- Hayward-Conjecture

Further applications, subset properties, distributed computing ...

Thanks!