Topological Graphs: Simple Drawings and Rotation Systems

Discrete and Computational Geometry
winter term 17/18
Motivation

*Graphs and Drawings of Graphs*

- **Up to now in this lecture:** Drawings of graphs with *straight-line* edges.
- **Next:** Drawings of graphs with more or less arbitrary *continuous curves* as edges.
Drawings

**Drawing (topological graph):** drawing of a simple graph in the plane or on the sphere, where ...

Vertices are distinct points.
Drawings

Drawing (topological graph): drawing of a simple graph in the plane or on the sphere, where ...

Vertices are distinct points.

Edges are continuous curves connecting two (end) points; edges do not pass through other vertices; no tangencies.

No three edges pass through a single crossing.
Simple Drawings

Simple / good drawing (simple top. graph): drawing of a simple graph in the plane or on the sphere, where...

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Any pair of edges intersects at most once: at a proper crossing or in a common end point.
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Simple Drawings

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**Question:** Can you simplify this?
Simple Drawings

Motivation: Drawings that minimize the number of crossings are simple.

Question: Possible without crossings?
Isomorphic Drawings

Two simple drawings are *isomorphic*, if they can be obtained from each other by a *homeomorphism* on the sphere.

⇔ all *vertex-edge-cell incidences are the same.*
Isomorphic Drawings

Two simple drawings are isomorphic, if they can be obtained from each other by a homeomorphism on the sphere.

⇔ all vertex-edge-cell incidences are the same.


[Kynčl 2009]: The number of isomorphism classes of simple drawings of $K_n$ is $2^{\Theta(n^4)}$. 
Weakly Isomorphic Drawings

Two simple drawings of $K_n$ are \textit{weakly isomorphic} if they have “the same intersections” i.e. \textit{same pairs of edges cross}.

Isomorphic drawings are also weakly isomorphic.
Weakly Isomorphic Drawings

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Different crossing number $\Rightarrow$ surely not weakly isomorphic.
Weakly Isomorphic Drawings

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All four drawings are weakly isomorphic.
Weakly Isomorphic Drawings

Two simple drawings of $K_n$ are weakly isomorphic if they have “the same intersections” i.e. same pairs of edges cross.

[Rafla 1988]: Enumeration of all weak isomorphism classes of $K_n$ for $n \leq 7$ under the assumption that every drawing contains a plane Hamiltonian cycle.

[Pach, Tóth 2004]: The number of weak isomorphism classes of simple drawings of $K_n$ is $2^{\Omega(n^2)}$ and $2^{O(n^2 \log n)}$.

[Kynčl 2013]: Improved upper bound of $2^{n^2\alpha(n)^O(1)}$. 

\[
\begin{array}{cccc}
\begin{array}{c}
\begin{tikzpicture}
\begin{scope}[scale=0.5]
\node (1) at (0,0) [circle,fill,inner sep=1pt] {1};
\node (2) at (1,1) [circle,fill,inner sep=1pt] {2};
\node (3) at (2,0) [circle,fill,inner sep=1pt] {3};
\node (4) at (1,-1) [circle,fill,inner sep=1pt] {4};
\node (5) at (0,2) [circle,fill,inner sep=1pt] {5};
\node (6) at (2,2) [circle,fill,inner sep=1pt] {6};
\foreach \i in {1,2,3,4,5,6}
\node [draw,circle,minimum size=0.5cm,fill,inner sep=1pt] at (\i) {};
\draw (1) -- (2) -- (3) -- (4) -- (5) -- (6) -- (1);
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& \neq \\
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$\Rightarrow$ Strong vs. weak isomorphism: $2^{\Theta(n^4)}$ vs. $2^{n^2 \alpha(n)^{O(1)}}$