What is a (mathematical) game?

- 2 players [A, B / L(eft), R(ight) / R(ed), B(lack)]
- the players move in turns (A, B, A, B, A, B …)
- Both players have complete information (no hidden cards …)
- No randomness (flipping coins, rolling dice …)
- A (finite) set of positions, one (or more) marked as starting position
What is a (mathematical) game, cont?

- For each position there exists a set of successors, (possibly empty)
- A player's move: transformation from one position to a legal successor
- Normal play: the first player which can NOT move loses (the other wins)
- Every game ends after a finite number of moves
- No draws

Chocolate game (Chomp)
Who wins a game?

- Which player wins the game (A,B)?
  - First player (starting) or second player?
  - Assume both players play optimal:
    There are 'First-Player-win' und 'Second-Player-win' games.
- What is the optimal strategy?

Chocolate game (again)

- Tweedledum-Tweedleddee-principle

Alice in Wonderland
by Lewis Carrol
NIM

- Who knows NIM?
  - n piles of \(k_1, \ldots , k_n > 0\) coins
- Valid moves:
  - chose a single (non-empty) pile
  - remove an arbitrary number of coins from the pile (at least one, at most all)
- Remember: normal play: the last one to make a valid move wins

Prime-game

- \(n\) integers \(f_1, \ldots , f_n > 1\)
- Valid move:
  - Choose a integer \(f_i > 1\)
  - Split \(f_i\) into (one or more) prime factors \(p_1, \ldots , p_k > 1\), \(k \geq 1\), and a rest \(f' > 1\).
    (i.e. \(f_i = p_1 \times \ldots \times p_k \times f'\))
  - Replace \(f_i\) with \(p_1, \ldots , p_k\) and \(f'\).
Poker-NIM

- Start position:
  - Same as for NIM

- Possible moves:
  - Similar to NIM, but instead of removing coins you may also put an arbitrary number of coins from your pool (built by previously taken coins) on a heap.

Northcott’s Game

- n×m chess board
  - one black, one white coin per row in different columns

- Valid move:
  - Chose a row
  - move the coin of your color left or right arbitrary many steps
  - don't jump over your opponents coin
Kayles (aka Rip Van Winkle’s Game)

- Bowling: Row of n pins. In a move hit one or two neighbored pins.

Dawson’s Kayles

- Bowling: Row of n pins. In a move always hit two neighbored pins. Single pins can be removed.
Kayles II

- Setting:
  - As for NIM
- Possible moves:
  - Chose an arbitrary, non-empty stack
  - Remove 1 or 2 coins from this stack
  - Optional: split the remaining stack into two non-empty, smaller stacks
  - Bowling: Row of n pins.
    In a move hit one or two neighbored pins.

Dawson’s Kayles II

- Setting:
  - As for NIM
- Possible moves:
  - Chose an arbitrary, non-empty stack
  - Remove 2 coins from this stack
  - Optional: split the remaining stack into two non-empty, smaller stacks
  - Bowling: Row of n pins.
    In a move hit two neighbored pins.
Monochromatic Triangle

- n points in the plane, general position
- Valid move:
  - Draw a line connecting two points, not crossing any other line
- The game ends when an empty triangle occurs

Triangulation Coloring Game

- Triangulation on n points in the plane, all edges are black
- Valid moves:
  - Select a black edge, color it green
- The game ends when the first green empty triangle occurs.
Which games?

- Games:
  - Chocolate game (chomp)
  - NIM
  - Prime-game
  - Poker NIM
  - Northcott’s Game
  - Kayles
  - Dawson’s Kayles
  - Monochromatic Triangle
  - Triangulation Coloring Game

Nimbers and NIM-Theory

- Nimbers \(*i\) are a 'code' for a game-position:
  - \(*i, i\neq 0 \Rightarrow 1^{st} \text{ player win (the next to move)}\)
  - \(*0 \Rightarrow 2^{nd} \text{ player win (the one just moved)}\)
  - \(\Rightarrow \text{ For optimal play, always try to reach a position with nimber } *0\)
A “good code” provides:

- From a \( *0 \) situation no legal move leads to another \( *0 \) situation
  - If I made a winning move, my opponent can not
- From any \( *i, i \neq 0 \), situation there is a legal move to a \( *0 \) situation
  - If my opponent gives me a non-optimal situation, I can make a winning move

MEX-rule (Minimal Excluded):

- The nimber of a position \( P \) is the smallest value which is NOT a nimber of any position which is reachable by a valid move from \( P \).
  - From a \( *0 \) situation no legal move leads to another \( *0 \) situation
  - From any \( *i, i \neq 0 \), situation there is a legal move to a \( *0 \) situation
Nimbers and NIM-Theory

- XOR-rule:
  - The nimber of set of positions is the XOR-sum of the nimber's of the situations.
  - \( \Rightarrow \) Simplifies computation of nimbers using the MEX-rule for several 'piles'

\[ \text{NIM:} \]

- A stack of size \( i \) has nimber \( *i \)
- The nimber of a group of stacks is their XOR-sum
- \( \Rightarrow \) Always make a position to get
  \[ *k_1 \otimes *k_2 \otimes *k_3 \otimes \ldots \otimes *k_n = 0 \]
Games, Triangulations, Theory

- Literature:
  - Winning Ways for Your Mathematical Plays
    E.R. Berlekamp, J.H. Conway and R.K. Guy:
  - Games on triangulations
    O. Aichholzer, D. Bremner, E.D. Demaine, F. Hurtado,
    E. Kranakis, H. Krasser, S. Ramaswami, S. Sethia,
    and J. Urrutia:

- links:

Thanks!