What makes a Tree a Straight Skeleton?*

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Abstract

Let $G$ be a cycle-free connected straight line graph with predefined edge lengths and fixed order of incident edges around each vertex. We address the problem of deciding whether there exists a simple polygon $P$ such that $G$ is the straight skeleton of $P$. We show that for given $G$ such a polygon $P$ might not exist, and if it exists it might not be unique. For small star graphs and caterpillars we give necessary and sufficient conditions for constructing $P$.

1 Introduction

The straight skeleton $S(P)$ of a simple polygon $P$ is a skeleton structure like Voronoi diagrams, but consists of straight-line segments only. Its definition is based on a so-called wavefront propagation process that corresponds to mitered offset curves. Each edge $e$ of $P$ emits a wavefront that moves with unit speed to the interior of $P$. Initially, the wavefront of $P$ consists of parallel copies of edges of $P$. However, during the wavefront propagation, topological changes occur: An edge event happens if a wavefront edge shrinks to zero length. A split event happens if a reflex wavefront vertex meets a wavefront edge and splits the wavefront into pieces, see Figure 1(right). The straight skeleton $S(P)$ is defined as the set of loci that are traced out by the wavefront vertices. The straight skeleton partitions $P$ into polygonal faces. Each face $f(e)$ belongs to a unique edge $e$ of $P$. Each straight skeleton edge belongs to two faces, say $f(e_1)$ and $f(e_2)$, and lies on the bisector of $e_1$ and $e_2$. Straight skeletons have many applications, like automatic roof construction, computation of mitered offset curves, and solving fold-and-cut problems. See [4] and Chapter 5.2 in [3] for further information and detailed definitions.

Although straight skeletons were introduced to computational geometry in 1995 by Aichholzer et al. [1], their roots actually go back to the 19th century. In textbooks about the construction of roofs (see e.g. [6], pages 86–122) using the angle bisectors (of the polygon defined by the ground walls) was suggested to design roofs where rainwater can run off in a controlled way. This construction is called Dachausmittlung and became rather popular. See [5] for related and partially more involved methods to obtain roofs from the ground plan of a house. In this book detailed explanations of the constructions and drawings of the resulting roofs can be found.

Maybe not surprisingly, none of this early works mentions the ambiguity of the non-algorithmic definition of the construction. It can be shown that the simple use of the bisector graph does not necessarily lead to a unique roof construction, and actually not even guarantees a plane partition of the interior of the defining boundary. See [1] for a detailed explanation and examples.

An interesting inverse problem was stated by Satyan L. Devadoss [2] and mentioned to us during CCCG 2011: Which graphs are the straight skeleton of some polygon? To give a more formal problem definition we denote with abstract geometric graph the set of combinatorial graphs, where the length of each edge and the cyclic order of incident edges around every vertex is predefined (and may not be altered). Let $G$ be the set of cycle-free connected abstract geometric graphs. Denote with $E(G)$ an embedding of $G \in G$ in the plane, that is, the vertices of $G$ are points in $\mathbb{R}^2$ and the edges of $G$ are straight line segments of the predefined length, connecting the corresponding points and respecting the predefined cyclic order of incident edges around each vertex. Further, denote with $P_{E(G)}$ the polygon resulting from con-

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necting the leaves of $G$ (with straight line segments) in cyclic order for the embedding $E(G)$. We call a simple polygon $P_{E(G)}$ suitable if its straight skeleton $S(P_{E(G)}) = E(G)$, for the embedding $E(G)$. If there exists a suitable polygon for a graph $G \in \mathcal{G}$, we call $G$ feasible, see Figure 1.

The obvious questions which arise from these definitions are: Which graphs $G \in \mathcal{G}$ are feasible? Are the suitable polygons for feasible graphs $G$ unique? How to construct a suitable polygon for a given graph $G$?

2 Star graphs

We start our discussion with the following simple fact on straight skeletons: All polygon edges whose straight skeleton faces contain a common vertex $u$ (of the straight skeleton) have equal orthogonal distance $t$ to $u$, because their wavefront edges reach $u$ at the same time $t$. That is, the supporting lines of those polygon edges are tangential to the circle with center $u$ and radius $t$.

Thus, in this section we consider a subset of $\mathcal{G}$, the so called star graphs. A star graph $S_n \in \mathcal{G}$, for $n \geq 3$ has $(n+1)$ vertices, one vertex $u$ with degree $n$ and $n$ leaves $v_1, \ldots, v_n$ ordered counter clockwise around $u$. The length of each edge $uv_i$, with $1 \leq i \leq n$, is denoted by $l_i$. W.l.o.g. let $l_1 = \max l_i$. Observe that the polygon $P_{E(S_n)}$ is star shaped and $v_i v_{i+1}$ (with $v_{n+k} := v_1 + (k-1) \mod n$) are its edges.

**Observation 1** If $S_n \in \mathcal{G}$ is a feasible star graph and $P_{E(S_n)}$ is a suitable polygon of $S_n$, then (1) all straight skeleton faces are triangles, (2) two consecutive vertices $v_i, v_{i+1}$ can not be both reflex, (3) $l_i < l_{i+1}$ for each reflex vertex $v_i$ of $P_{E(S_n)}$, and (4) all edges of $P_{E(S_n)}$ have equal orthogonal distance $t$ to $u$, with $t \in (0, \min l_i)$.

As a given $S_n \in \mathcal{G}$ is possibly not feasible and a suitable polygon may not be known or does not exist, we define a polyline $L_{S_n}(t, A)$: The vertices $v_1, \ldots, v_{n+1}$ of $L_{S_n}(t, A)$ are the leaves, $v_1, \ldots, v_n$ of $S_n$, in the same order as for $S_n$, and one additional vertex $v_{n+1}$ succeeding $v_n$. The vertices $v_1, \ldots, v_n, v_{n+1}$ have the corresponding distances (predefined in $S_n$) $l_1, \ldots, l_n, l_1$ to $u$. $A$ is an assignment for each vertex whether it should be convex or reflex, as seen from $u$. As $l_1 = \max l_i$, $v_1$ and $v_{n+1}$ are always convex (fact (3) in Observation 1). For the remaining vertices any convex/reflex assignment, which respects the facts (2) and (3) in Observation 1, can be considered. The edges of $L_{S_n}(t, A)$ have equal orthogonal distance $t$ to $u$. Of course, not all possible combinations of $t$ and an arbitrary embedding $E(S_n)$ allow such a polyline. But it is possible to construct $L_{S_n}(t, A)$ and $E(S_n)$ simultaneously for a fixed $t \in (0, \min l_i)$.

For a fixed assignment $A$ and a fixed $t \in (0, \min l_i]$ we construct $L_{S_n}(t, A)$ (and $E(S_n)$) in the following way. Consider the circle $C$ with center $u$ and radius $t$. Start with $v_1$ at polar coordinate $(l_1, 0)$, with $u$ as origin. For each $v_i$, $i = 2 \ldots (n+1)$, consider a tangent $g_{i-1}$ to $C$ (such that the vertices will be placed counter clockwise around the circle) through $v_{i-1}$. If $v_{i-1}$ is convex, then there exist two points with distance $l_i$ ($l_i$ for $v_{n+1}$) on $g_i$. If $v_i$ is assigned to be reflex, then $v_i$ is placed on the point closer to $v_{i-1}$, and if $v_i$ is assigned to be convex, then $v_i$ is placed on the other point. If $v_{i-1}$ is reflex, then there exists only one applicable point for placing $v_i$ on $g_{i-1}$. See Figure 2.

The $L_{S_n}(t, A)$ constructed this way is unique (for fixed $t$ and $A$), and may be not simple (e.g. when circling $C$ many times), simple but not closed ($v_n v_1 \neq v_1$), or simple and closed ($v_n v_1 \equiv v_1$). In the latter case, the construction reveals a witness pair $(t, A)$ for the existence of some $E(S_n)$, a suitable polygon $P_{E(S_n)}$, and thus the feasibility of $S_n$.

It is easy to see, that for each suitable polygon $P_{E(S_n)}$, there exists a polyline $L_{S_n}(t, A)$ (just duplicate the vertex $v_1$). Hence, deciding feasibility of $S_n$ is equivalent to finding an assignment $A$ and a $t \in (0, \min l_i]$ such that $L_{S_n}(t, A)$ is closed and simple. For a polyline $L_{S_n}(t, A)$ and a corresponding embedding $E(S_n)$, we denote with $\alpha_i$, $i = 1 \ldots n$, the counter clockwise angle at $u$, spanned by $uv_i$ and $ur_{i+1}$. (Note that for a suitable polygon $P_{E(S_n)}$ $\alpha_i$ can be defined the same way, with $v_{n+1} \equiv v_1$.) It is easy to see that the sum of all $\alpha_i$ is $2\pi$ if and only if $L_{S_n}(t, A)$ is closed and simple.

**Lemma 1** Let $S_n \in \mathcal{G}$, distance $t \in (0, \min l_i]$ and assignment $A$ be fixed, and let $L_{S_n}(t, A)$ be the resulting polyline. Then $\alpha_A(t) := \sum_{i=1}^n \alpha_i$ can be expressed as

$$\alpha_A(t) = 2 \sum_{i=1}^n \begin{cases} \arccos \frac{t}{l_i} & \text{if } v_i \text{ convex} \\ \pi - \arccos \frac{t}{l_i} & \text{if } v_i \text{ reflex} \end{cases}.$$ 

(1)

**Proof.** Omitted in this version. \qed

For the following result we use the first derivative of $\alpha_A$:

$$\alpha_A'(t) = 2 \sum_{i=1}^n \begin{cases} \frac{1}{l_i^2} & \text{if } v_i \text{ convex} \\ \frac{1}{\sqrt{l_i^2 - t^2}} & \text{if } v_i \text{ reflex} \end{cases} - 2 \sum_{i=1}^n \frac{1}{\sqrt{l_i^2 - t^2}}.$$ 

(2)
Lemma 2 A suitable convex polygon for a star graph $S_n$ exists if and only if $\sum \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If a suitable convex polygon exists then it is unique.

Proof. As all vertices are assumed to be convex, we obtain $\alpha_A(0) = n\pi > 2\pi$. Furthermore, we observe that $\alpha_A(t)$ is monotonically decreasing since $\alpha_A'(t) < 0$ for all $t \in (0, \min_i l_i]$. Hence, there is a $t \in (0, \min_i l_i]$ with $\alpha(t) = 2\pi$ and only if $\alpha_A(0) = \alpha_A(t) = 2\pi$ which is $\sum \arccos \frac{\min_i l_i}{l_i} \leq \pi$. If this is the case the solution is unique as $\alpha(t)$ is monotonic. \qed

For $n = 3$, $\alpha_A(0) = 3\pi$ and $\alpha_A(\min_i l_i) < 2\pi$, and thus we immediately get the following corollary.

Corollary 3 For every $S_3$ there exists a unique suitable convex polygon.

Considering star graphs with $n = 5$, we show in the following lemma that they are not always feasible, and that suitable polygons (if they exist) are not always unique.

Lemma 4 There exist infeasible star graphs, $S_5 \in \mathcal{G}$. Further, there exist feasible star graphs for which multiple suitable polygons exist.

Proof. To prove the first claim consider a star graph with $n = 5$, $l_1 = l_2 = l_3 = l_4 = 1$, and $l_5 = 0.25$. There exist only two possible assignments: either all vertices convex or all but $v_5$ convex. It is easy to check that for both assignments $\sum \alpha_i > 2\pi$, for every $t \in (0, \min_i l_i]$. To prove the second claim consider a star graph with $n = 5$, $l_1 = l_2 = 1$, $l_3 = 0.6$, $l_4 = 0.79$, and $l_5 = 0.75$. Assign all vertices convex, except for $v_2$. Then $\sum \alpha_i$ evaluates to $2\pi$ for $t \approx 0.537$ and $t \approx 0.598$. Hence, there exist (at least) two different suitable polygons for this star graph. \qed

In the following we discuss sufficient and necessary conditions for the feasibility of a star graph $S_4$. By Lemma 2 we know in which cases suitable convex polygons exist. The remaining cases are solved by the following lemma.

Lemma 5 Consider an $S_4$ for which no suitable convex polygon exists. A suitable non-convex polygon exists if and only if $\frac{1}{\min_i l_i} < \sum_{j=1}^{4} \frac{1}{\min_i l_i}$. \[ \begin{array}{c}
\frac{1}{\sqrt{l_1^2 - t^2}} > \frac{3}{\sqrt{l_1^2 - t^2}} \quad \Leftrightarrow \quad 1 > \frac{3}{\sum_{i=1}^{3} 1 - \frac{l_i^2 - t^2}{l_i^2 - l_j^2}}
\end{array} \]

The right side of this equivalence is true since

$$1 \geq \frac{3}{\sum_{i=1}^{3} \sqrt{1 - \frac{l_i^2 - t^2}{l_i^2 - l_j^2}}} \geq \frac{3}{\sum_{i=1}^{3} 1 - \frac{l_i^2 - l_j^2}{l_i^2 - l_j^2}},$$

where the first inequality is given by $\alpha_A(0) > 0$ and the second inequality holds for all $t \in (0, l_j)$. To conclude, we have shown that if no suitable convex polygon exists for some $S_4$, then a suitable non-convex polygon exists for this $S_4$ if and only if $\alpha'(0) < 0$, which is equivalent to $\frac{1}{\min_i l_i} < \sum_{j=1}^{3} \frac{1}{l_j}$, as claimed in the lemma.

3 Caterpillar graphs

The techniques developed in the previous section can be generalized to so-called caterpillar graphs. A caterpillar graph $G \in \mathcal{G}$ is a graph that becomes a path if all its leaves (and their incident edges) are removed. We call this path the backbone of $G$. Figure 1 shows a caterpillar graph whose backbone comprises three backbone edges.

In general, a caterpillar graph has $m$ backbone vertices, consecutively denoted by $v^1_0, \ldots, v^m_0$. We denote the adjacent vertices of a backbone vertex $v^0_0$, with $i$ incident edges, by $v^1_0, \ldots, v^i_0$, such that $v^i_0 = v^{i+1}_0$ for $1 \leq i < m$. Furthermore, we denote by $l_i$ the length of the edge $v^i_0 v^{i+1}_0$, see Figure 3. Let us consider a polygon $P$ whose straight skeleton $S(P)$ forms a caterpillar graph.

Observation 2 All edges of $P$ whose straight skeleton faces contain the same backbone vertex $v^0_0$ have identical orthogonal distance to $v^0_0$. We denote this orthogonal distance by $r_i$. Hence, the supporting lines of the corresponding polygon edges are tangents to the circle of radius $r_i$ centered at $v^0_0$, see Figure 3.
Lemma 6 The radii $r_2, \ldots, r_m$ of a suitable polygon $P_{E(G)}$ for some given caterpillar graph $G$ are determined by $r_1$ and the predefined edge lengths of $G$ according to the following recursions, for $1 \leq i < m$:

\begin{align*}
r_{i+1} &= r_i + \frac{l_i}{2} \sin \beta_i \\
\beta_i &= \beta_{i-1} + (1 - k_i/2)\pi + \frac{k_i-1}{2} \arcsin \frac{r_j}{l_j} \\
&\quad \quad \text{if } v_j \text{ is convex} \\
&\quad \quad \text{or } \pi - \arcsin \frac{r_j}{l_j} \\
&\quad \quad \text{if } v_j \text{ is reflex}
\end{align*}

For $i = 1$ we define that $\beta_0 = 0$ and $v_1 \neq v_0$ being true for all $1 \leq j < k_1$.

Proof. Denote with $e$ one of the two edges of $P_{E(G)}$ whose faces of $S(P_{E(G)})$ contain the edge $v_0 v_{i+1}$. The supporting line of $e$ is tangential to the circles $v_0$ and $v_{i+1}$. Considering the shaded right-angled triangle in Figure 3, we obtain $r_{i+1} = r_i + \frac{l_i}{2} \sin \beta_i$.

Considering the polygon $P_i'$ (bold dashed in Figure 3) which comprises the edges of $P_{E(G)}$ whose faces of $S(P_{E(G)})$ contain $v_0$, trimmed by two additional edges orthogonal to $v_0 v_i v_{i+1}$ and $v_0 v_{i+1}$, respectively. $P_i'$ comprises $k_i+2$ vertices ($k_i+1$ for $P_i'$) and hence, the sum of inner angles equals $k_i \pi ((k_i-1)\pi$ for $P_i'$). On the other hand, we can express this sum as follows (also for $P_i'$), which implies the second recursion:

\[
k_i \pi = 2\pi + 2\beta_{i-1} - 2\beta_i + 2 \sum_{j=1}^{k_i-1} \begin{cases} \arcsin \frac{r_j}{l_j} & v_j \text{ is convex} \\ \pi - \arcsin \frac{r_j}{l_j} & v_j \text{ is reflex} \end{cases}
\]

Corollary 7 The sum of the inner angles of $P_{E(G)}$ with convexity assignment $A$ is a function

\[
\alpha_A(r_1) = 2 \sum_{j=1}^{n} \begin{cases} \arcsin \frac{r_j}{l_j} & v_j \text{ is convex} \\ \pi - \arcsin \frac{r_j}{l_j} & v_j \text{ is reflex} \end{cases},
\]

where $r_v$ denotes the radius of the circle at the backbone vertex that is adjacent to $v_j$ and $l_j$ denotes the length of the incident edge of $G$.

An Inversion of the Straight Skeleton Problem

What makes a Tree a Straight Skeleton?

Which tree is a Straight Skeleton?