On semi-simple drawings of the complete graph∗

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Abstract

In this work we study rotation systems and semi-simple drawings of $K_n$. A simple drawing of a graph is a drawing in which every pair of edges intersects in at most one point. In a semi-simple drawing, edge pairs might intersect in multiple points, but incident edges only intersect in their common endpoint. A rotation system is called (semi-)realizable if it can be realized with a (semi-)simple drawing. It is known that a rotation system is realizable if and only if all its 5-tuples are realizable. For the problem of characterizing semi-realizability, we present a rotation system with six vertices that is not semi-realizable, although all its 5-tuples are semi-realizable. Moreover, by an exhaustive computer search, we show that for seven vertices there exist minimal not semi-realizable rotation systems (that is, rotation systems in which all proper sub-rotation systems are semi-realizable). This indicates that checking semi-realizability is harder than checking realizability. Finally we show that for semi-simple drawings, generalizations of Conway’s Thrackle Conjecture and the conjecture on the existence of plane Hamiltonian cycles do not hold.

1 Introduction

In a drawing of a graph, vertices are represented by distinct points in the plane, and edges are represented by Jordan arcs with their vertices as endpoints. Additionally, edges must not contain any other vertices, no three edges intersect in the same point, and any intersection between two edges is either a proper crossing or a common endpoint. In a simple drawing of a graph any two edges intersect at most once.

Two simple drawings $G$ and $H$ are weakly isomorphic if there exists an incidence-preserving bijection between their vertices, such that two edges of $G$ cross if and only if the corresponding two edges of $H$ cross. We will consider the classes of weakly isomorphic simple drawings of $K_n$, the complete graph on $n$ vertices. We can efficiently handle them using rotation systems. The rotation of a vertex in a drawing is the cyclic order of all edges incident to it. The rotations of all vertices of a drawing form its rotation system. A rotation system is said to be realizable if it is a rotation system of a simple drawing. Two simple drawings of $K_n$ are weakly isomorphic if and only if they have the same rotation system (up to reflection) [5].

A semi-simple drawing of a graph is a drawing in which edges that share a vertex do not cross, but edges not sharing a vertex are allowed to (properly) cross an arbitrary number of times; see [9] and also [2]. Semi-simple drawings can have regions that are bounded only by two continuous pieces of (two) edges, which we denote as lenses. If a lens contains no vertex, we call it empty. We call a rotation system semi-realizable if it can be drawn as a semi-simple drawing. It is known that every semi-realizable rotation system can be drawn without empty lenses [5]. We call semi-simple drawings without empty lenses minimal.

Motivated by the amount of structure of simple drawings determined by rotation systems, we investigate the properties of (minimal) semi-simple drawings w.r.t. their rotation systems. As for simple drawings, a semi-simple drawing with three vertices can only be a simple 3-cycle. A simple drawing of $K_4$ has either no crossing or one crossing; a semi-simple drawing with the latter rotation system has exactly one pair of edges that crosses an odd number of times. We observe that for $K_4$, all minimal semi-simple drawings are simple. For $K_n$, the rotation system of a semi-simple drawing determines whether two edges cross an even or an odd number of times. This is related to the Hanani-Tutte theorem, which states that a graph is planar if and only if in a drawing any two non-adjacent edges cross an even number of times, and which motivates the notion of the independent odd crossing number [11]. While the Hanani-Tutte theorem (like probably most results considering the odd crossing number) considers all drawings of a given
graph, we are interested in drawings of $K_n$ with a
given rotation system.

In this paper, we aim for determining semi-
realizability of rotation systems. To decide whether
a given rotation system is semi-realizable, we use a
backtracking approach based on an algorithm for gen-
erating simple drawings [1]. To adapt it for semi-
simple drawings we use an upper bound on the num-er of times two edges can cross in a minimal semi-
simple drawing, i.e., without creating empty lenses.

Given a rotation system of $K_n$, we call the rotation
systems restricted to 4 or 5 elements 4-tuples and 5-
tuples, respectively. A rotation system is realizable if
and only if all its 4- and 5-tuples are realizable [6].
As a 4-tuple is semi-realizable if and only if it is re-
alizable, it was conjectured that realizability of all
4-tuples is sufficient for semi-realizability. We refute
this conjecture in Section 3.

2 Realizability of rotation systems

To decide the realizability of a rotation system, it suf-
fices to check that all 5-tuples (and thus 4-tuples) are
realizable [6]. While a brute-force approach to check
all the 4-tuples yields an $O(n^3)$ time algorithm, we
show that only $O(n^2)$ checks are needed.

Observation 1 In a realizable 4-tuple, the rotations
of three of its vertices determine the rotation system.

Lemma 2 In any rotation system of five vertices, the
number of non-realizable 4-tuples is even.

Proof. A flip in a rotation system is the exchange of
the positions of two neighboring vertices in the rota-
tion of a vertex. Let $R$ be a rotation system of the
vertices $\{a, b, c, d, e\}$ and consider an arbitrary flip of
two neighboring vertices $b$ and $c$ in the rotation of $a$.
This flip only affects the two 4-tuples on $\{a, b, c, d\}$
and on $\{a, b, c, e\}$. Since in each of these two 4-tuples
only one rotation changes, the realizability of the two
4-tuples switches due to Observation 1. Hence, by
one single flip, the number of non-realizable 4-tuples
in $R$ either changes by two or stays the same. Using
the argument of various sorting algorithms, we know
that by multiple single flips we can obtain every pos-
sible rotation system. Since we know that a 5-tuple
without a non-realizable 4-tuple exists the number of
non-realizable 4-tuples in a 5-tuple is always even.

Lemma 3 Let $R$ be a rotation system of $n$ vertices.
If $R$ contains a non-realizable 4-tuple, then every
vertex of $R$ is contained in a non-realizable 4-tuple.

Proof. Assume that there are four vertices $v, x, y, z$
whose sub-rotation system is non-realizable. Then a
5-tuple $\{u, v, x, y, z\}$ with any fifth vertex $u$ is also
non-realizable. From Lemma 2 we know that there

exists a second non-realizable 4-tuple in this 5-tuple.
This 4-tuple must include $u$. $\square$

Thus, to decide realizability of all 4-tuples it suffices
to check all 4-tuples containing an arbitrary vertex $u$.

Corollary 4 Checking the realizability of all 4-tuples
in a rotation system can be done in cubic time.

There are five realizable 5-tuples and two other
semi-realizable ones (see the black sub-drawings in
Figure 1). While for 4-tuples it was sufficient to check
all those containing one fixed element, this approach
no longer works for realizability of 5-tuples, as there
are arbitrarily large semi-simple drawings containing
only one non-realizable 5-tuple. Such drawings can
for example be constructed modifying the drawings
in Figure 1 by adding an arbitrary number of vertices
similar to the red one.

3 Semi-realizability of rotation systems

From the computations we know that all rotation sys-
tems of five vertices that have only realizable 4-tuples
are either realizable or semi-realizable; see Section 5.
Moreover, each of them has a unique minimal semi-
simple drawing (up to homeomorphism of the sphere).
However, in general it is not the case that realizability
of 4-tuples implies semi-realizability, which disproves
the related conjecture mentioned in the introduction.

Theorem 5 The semi-realizability of a rotation sys-
tem does not follow from the realizability of all its
4-tuples.

Figure 2 shows two partial drawings of a rotation
system with six vertices that is not semi-realizable,
but whose 5-tuples are all semi-realizable. The left
one is equivalent to the rotation system of the geo-
metric drawing of $K_5$ with the five vertices in convex
position; all other 5-tuples are equivalent to the rota-
tion system of the semi-simple drawing to the right.

For proving that the given example in Figure 2 is
not semi-realizable, we argue about area-containment

Figure 1: Semi-simple drawings extending a non-
realizable 5-tuple (depicted in black).
Lemma 6 For a rotation system of a semi-simple drawing of $K_n$ and a 3-cycle through vertices $a$, $b$, and $c$ of this drawing, consider the two regions bounded by the cycle. Any other vertex $v$ must lie in the region in which at least two of the three edges to $v$ emanate from $a$, $b$, and $c$.

The statement of Lemma 6 can be proven in a way similar to the one for simple drawings. The proofs of Lemma 6 and Theorem 5 are deferred to the full version. We remark that the statement of Theorem 5 also follows from the computations; see Section 5.

4 Computational issues

To decide algorithmically whether a rotation system is semi-realizable, we used a backtracking approach based on the algorithm for realizing simple drawings used in [1]. We modified it allowing multiple proper crossings per edge pair. It thus requires an upper bound on the maximum number of proper crossings per edge pair in a minimal semi-simple drawing of $K_n$. From computations we get that 5 and 10 crossings are such upper bounds for $n = 6$ and $n = 7$, respectively.

Using these parameters, we verified that the example in Figure 2 is the only non-semi-realizable one with six vertices where all 4-tuples are realizable. For seven vertices we exhaustively analyzed all possible rotation systems. We found 480 non-semi-realizable rotation systems such that the sub-rotation system of every proper subset of vertices is semi-realizable.

To determine an upper bound on the maximum number of crossings per edge pair, we use another backtracking algorithm. It enumerates all different ways how two edges can cross multiple times without creating some forbidden patterns.

We proceed with an overview of how the algorithm works; a detailed description can be found in [1].

The drawings with two edges in which we are interested are sub-drawings of minimal semi-simple drawings of $K_n$, i.e., we want them to be completable to a minimal semi-simple drawing of $K_n$. In particular, the configuration shown in Figure 3, which we call a spiral, is forbidden, as it is not completable: vertex $x$ lies inside a lens bounded by the edges $ab$ and $xy$ (gray region in Figure 3), and thus, edges $xa$ and $xb$ cannot be added to the drawing.

The algorithm starts with a single edge. We also fix the starting vertex of the second edge. In every step the drawing is extended by adding a crossing (and maybe a vertex) in all possible ways avoiding spirals and empty lenses. It ends when we are forced to create an empty lens for which we do not have a point remaining to place in.

With this algorithm we can compute an upper bound on the maximal number of crossings of an edge pair in a semi-simple drawing of $K_n$. There exits a construction for a spiral-free drawing of two edges with $2n^2 - 4$ crossings for $n$ vertices [1]; we elaborate on this in more detail in the full version.

5 Results from computations

Table 1 summarizes the results obtained from the computations described in Section 4. We started with all rotation systems entirely consisting of realizable 4-tuples, and subtracted the realizable ones; see [1].

For the remaining sets we checked whether they can be drawn semi-simple with a predefined maximal number of crossings per edge pair. Increasing this maximal number stepwise from 2 to the maximum, we obtained semi-simple drawings with the minimum maximal number of crossings per edge pair. The number of rotation systems requiring 2, 3, 4 or more crossings is also given in Table 1. For $n = 6$ there is precisely one rotation system which cannot be drawn semi-simple. For $n = 7$ there are 340 rotation sys-
Conway’s thrackle conjecture. A thrackle is a simple drawing of a graph where each pair of edges either shares an endpoint or crosses exactly once \cite{7}. Conway conjectured that the number of edges of a thrackle cannot exceed the number of its vertices (see \cite{12}). This conjecture implies that no simple drawing of $K_n$ can contain a thrackle with $n+1$ edges. In Figure 5 (right) we show a semi-simple drawing of $K_7$ that contains a subgraph with eight edges that pairwise either share a common vertex or cross an odd number of times. This shows that a generalization of Conway’s thrackle conjecture does not hold for semi-simple drawings. See e.g. \cite{3} for related but different generalizations of the thrackle conjecture.

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References