Lower bounds for the number of small convex $k$-holes*

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Abstract

Let $S$ be a set of $n$ points in the plane in general position, that is, no three points of $S$ are on a line. We consider an Erdős-type question on the least number $h_k(n)$ of convex $k$-holes in $S$, and give improved lower bounds on $h_k(n)$, for $3 \leq k \leq 5$. Specifically, we show that $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$, $h_4(n) \geq \frac{n^2}{4} - \frac{9n}{4} - o(n)$, and $h_5(n) \geq \frac{3n}{4} - o(n)$.

1 Introduction

Let $S$ be a set of $n$ points in the plane in general position, that is, no three points of $S$ lie on a common straight line. A $k$-hole of $S$ is a simple polygon, $P$, spanned by $k$ points from $S$, such that no other point of $S$ is contained in the interior of $P$. A classical existence question raised by Erdős [8] is: “What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex $k$-hole?”. Escher Klein observed that every set of 5 points contains a convex 4-hole, and Harborth [12] showed that every set of 10 points determines a convex 5-hole. Both bounds are tight w.r.t. the cardinality of $S$. Only in 2007/08 Nicolás [14] and independently Gerken [11] proved that every sufficiently large point set contains a convex 6-hole. On the other hand, Horton [13] showed that there exist arbitrarily large sets which do not contain any convex 7-hole; see [1] for a brief survey.

A generalization of Erdős’ question is: “What is the least number $h_k(n)$ of convex $k$-holes determined by any set of $n$ points in the plane?” In this paper we concentrate on this question for $3 \leq k \leq 5$, that is, the number of empty triangles (3-holes), convex 4-holes, and convex 5-holes. We denote by $h_k(S)$ the number of convex $k$-holes determined by $S$, and by $h_k(n) = \min_{|S|=n} h_k(S)$ the number of convex $k$-holes any set of $n$ points in general position must have. Throughout this paper let $\log x = \frac{\log x}{\log 2}$ be the binary logarithm. Furthermore, we denote with $CH(S)$ the convex hull of $S$ and with $\partial CH(S)$ the boundary of $CH(S)$.

We start in Section 2 by providing improved bounds on the number of convex 5-holes. In particular, increasing the so far best bound $h_5(n) \geq \frac{n}{2} - O(1)$ [16] to $h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875$. In Section 3 we combine these results with a technique recently introduced by García [9, 10], and improve the currently best bounds on the number of empty triangles and convex 4-holes, $h_3(n) \geq n^2 - \frac{372n}{7} + \frac{22}{7}$ and $h_4(n) \geq n^2 - \frac{14n}{3} - \frac{9}{4}$ (both in [10]), to $h_3(n) \geq n^2 - \frac{12n}{7} + \frac{22}{7}$ and $h_4(n) \geq n^2 - \frac{9n}{4} - 1.2641 n^{0.926} + \frac{199}{27}$, respectively.

2 Convex 5-holes

The currently best upper bound on the number of convex 5-holes, $h_5(n) \leq 1.0207n^2 + o(n^2)$ is by Bárány and Valtr [5], and it is widely conjectured that $h_5(n)$ grows quadratically. Still, to this date not even a super-linear lower bound is known.

As early as in 1987 Dehnhardt presented a lower bound of $h_5(n) \geq 3\left\lceil \frac{n}{12} \right\rceil$ in his thesis [6]. Unfortunately, this result, published in German only, remained unknown to the scientific community until recently. Thus, the best known lower bound was $h_5(n) \geq \left\lceil \frac{n^2}{6} \right\rceil$, obtained by Bárány and Károlyi [4]. In the presentation of [9] this bound was improved to $h_5(n) \geq \frac{5}{4}n - \frac{9}{4}$. A slightly better bound $h_5(n) \geq 3\left\lceil \frac{n^2}{6} \right\rceil$ was presented in [2], which was then sharpened to $h_5(n) \geq \left\lceil \frac{3}{4} (n - 11) \right\rceil$ in [3]. The latest and so far best bound of $h_5(n) \geq \frac{n}{2} - O(1)$ is due to Valtr [16]. In this section we further improve this bound to $h_5(n) \geq \frac{3n}{4} - o(n)$.

We start by fine-tuning the proof from [3], showing $h_5(n) \geq \left\lceil \frac{3}{4} (n - 11) \right\rceil$, by utilizing the results $h_5(10) = 1$ [12], $h_5(11) = 2$ [6], and $h_5(12) \geq 3$ [6]. Although this does not lead to an improved lower bound of $h_5(n)$ for large $n$, it provides better lower bounds for small values of $n$; see Table 1.

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Lemma 1 Every set $S$ of $n$ points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1,2,3\}$) contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Proof. Because of $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) \geq 3$ this is true for $m = 0$. Obviously $h_5(n) \geq h_5(n-1)$. Hence, $h_5(n) \geq 3$ for any $n \geq 12$.

If there exists a point $p \in ((\partial CH(S)) \cap S)$ that is a point of a convex 5-hole, then $h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1)$. In this case, the lemma is true by induction, as for $t = 1$ and $m > 0$, $h_5(n-1) = h_5(7 \cdot m + 9) \geq h_5(7 \cdot (m-1) + 9 + 3)$. (The case $t \in \{2,3\}$ is trivial.)

Otherwise, each point $p \in ((\partial CH(S)) \cap S)$ is not a point of a convex 5-hole. For $m > 0$ choose one such point $p$ (e.g. the bottom-most one) and partition $S \setminus \{p\}$ (in clockwise order around $p$) into the following successive disjoint subsets: $S_0$ containing the first 7 points; $S'_0$ containing the next 4 points; $(m-1)$ pairs of subsets: $S_i$ containing 3 points and $S'_i$ containing 4 points $(1 \leq i \leq (m-1))$; and the subset $S_{rem}$ containing the remaining $(t+4)$ points. See Figure 1 for a sketch.

![Figure 1: Partition of $S \setminus \{p\}$ clockwise around an extreme point $p$](image)

Figure 1: Partition of $S \setminus \{p\}$ clockwise around an extreme point $p$: starting with the pair $S_0, S'_0$; continuing with $(m-1)$ pairs of sets $S_i, S'_i$, for $1 \leq i \leq (m-1)$, with $|S_i| = 3$ and $|S'_i| = 4$; and ending with the remainder set $S_{rem}$. The subset $S_0 \cup S'_0 \cup \{p\}$ has cardinality 12 and thus contains at least 3 convex 5-holes. The same is true for each subset $S'_{i-1} \cup S_i \cup S'_{i+1} \cup \{p\}$ $(1 \leq i \leq (m-1))$. Finally, the subset $S_{rem} \cup \{p\}$ has cardinality $(9 + t)$ and therefore contains at least $t$ convex 5-holes. Note that we count every convex 5-hole at most once, as the considered subsets of 10, 11, and 12 points, respectively, overlap in at most 4 points. In total this gives at least $3 + (m-1) \cdot 3 + t = 3 \cdot \frac{n-9-t}{7} + t = \frac{3n-27+4t}{7}$ convex 5-holes.

$$
\begin{array}{c|cccccccccccc}
 h_5(n) & 1 & 2 & 3 & 3.4 & 3.6 & 3.9 & \geq 3 & \geq 4 & \geq 5 & \geq 6 & \geq 6 & \geq 7 & \geq 8 & \geq 9 & \geq 9 & \geq 10 \\
\end{array}
$$

Table 1: The updated bounds on $h_5(n)$ for small values of $n$.

Corollary 2 Every set $S$ of 17 points in the plane in general position contains at least $h_5(17) \geq 4$ convex 5-holes.

Table 1 shows the bounds on $h_5(n)$ obtained by Lemma 1, for some small values of $n$. By Harborth [12] $h_5(10) = 1$, and by Dehnhardt [6] $h_5(11) = 2$ and $h_5(12) \geq 3$. The bounds for $n = 51$ and for $57 \leq n < 62250$ (not shown in the table) are due to $h_5(n) \geq \frac{n^3}{2} - 7$ from Valtr [16]. The bounds $h_5(12) \leq 3$, $h_5(13) \leq 4$, $h_5(14) \leq 6$, and $h_5(15) \leq 9$ are from [3, 17].

In the following theorem we present an improved lower bound on $h_5(n)$ for larger $n$.

Theorem 3 Every set $S$ of $n \geq 12$ points in the plane in general position contains at least $h_5(n) \geq \frac{3n}{4} - \frac{n^4}{16} - \frac{15}{8} = \frac{3n}{4} - o(n)$ convex 5-holes.

Proof. For $12 \leq n < 17$ we count three convex 5-holes for $S$. For $17 \leq n < 24$ we can count four convex 5-holes for $S$ by Corollary 2.

If $n \geq 24$ consider an (almost) halving line $\ell$ of $S$ which splits $S$ into $S_L \cup \{S_L = \left\lfloor \frac{n}{2} \right\rfloor \}$ and $S_R \cup \{S_R = \left\lfloor \frac{n}{2} \right\rfloor \}$ and does not contain any point of $S$. See Figure 2.

![Figure 2: A point set $S$ split by a halving line $\ell$ into two point sets $S_L, S_R \subset S$. The line $\ell'$ cuts off a set $S' \subset S$, consisting of 8 points of $S_L$ and 4 points of $S_R$. The line $\ell''$ is parallel to $\ell'$ and halves $S_L \cap S'$. Furthermore, consider a line $\ell''$ that intersects $\ell$ and cuts off a set $S'' \subset S$, consisting of eight points from $S_L$.](image)
and four points from \( S_R \). That this is in fact possible is folklore, see e.g. Exercise 4.5 (b) in [7]. Let a line \( \ell' \) be parallel to \( \ell \) and split \( S' \cap S_L \) into two groups of four points, and let \( S'' \subset S' \) be the set which is cut off by \( \ell'' \). Note that neither \( \ell' \) nor \( \ell'' \) contain any points of \( S \).

As \(|S'| = 12\) we have that \( S' \) contains at least three convex 5-holes. We distinguish two cases.

**Case 1:** \( S' \) contains at least three convex 5-holes which are not intersected by \( \ell \). Then each of these 5-holes contains only points from \( S_L \) and thus at least one point above \( \ell'' \). We count the three convex 5-holes for the set \( S_L \) and continue on \( S \setminus S'' \).

**Case 2:** \( S'' \) contains at most two convex 5-holes which are not intersected by \( \ell \). Then at least one convex 5-hole in \( S' \) is intersected by \( \ell \). We count one convex 5-hole for the halving line \( \ell \) and continue on \( S \setminus S'' \).

Note that in both cases we cut off at least four points from \( S_L \), but at most four points from \( S_R \). Thus, we can repeat this process until we have processed all \( \left\lfloor \frac{n}{2} \right\rfloor \) points of \( S_L \). Let \( c_L \) be the number of convex 5-holes counted for \( \ell \) when processing \( S_L \). Hence, Case 2 appeared \( c_L \) times, and Case 1 appeared at least \( \left\lfloor \frac{1}{4} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_L \right) \) - 1 times. Therefore, the number of convex 5-holes we counted in \( S_L \) (i.e., not intersecting \( \ell \)) is \( h_5(S_L) \geq 3 \left( \left\lfloor \frac{1}{4} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_L \right) - 1 \right) \).

Repeating the same procedure for \( S_R \) (excluding the roles of \( S_L \) and \( S_R \)), we obtain \( h_5(S_R) \geq 3 \left( \left\lfloor \frac{1}{4} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) - 1 \right) \), where \( c_R \) is the number of convex 5-holes which we counted for \( \ell \) when processing \( S_R \). Note that any convex 5-hole intersected by \( \ell \), which we counted while processing \( S_L \), might have occurred again when processing \( S_R \). Thus, the total number \( c \) of convex 5-holes intersected by \( \ell \) is at least \( \max \{c_L, c_R\} \geq \frac{c_L + c_R}{2} \).

As \( h_5(S) = h_5(S_L) + h_5(S_R) + c \), we obtain

\[
h_5(S) \geq 3 \cdot \left( \left\lfloor \frac{1}{4} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_L \right) - 1 \right) + 3 \cdot \left( \left\lfloor \frac{1}{4} \right\rfloor \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) - 1 \right) + \frac{c_L + c_R}{2}.
\]

Considering

\[
\left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{4} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{n}{2} \right\rfloor}{4} \right\rfloor = \begin{cases} 2 \cdot \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \text{ is even} \\ 2 \cdot \left\lfloor \frac{n}{4} \right\rfloor + \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}
\]

is \( \geq \frac{n}{4} - \frac{6}{4} \) in both cases, careful transformation gives

\[
h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2}
\]

as a first lower bound for the number of convex 5-holes in \( S \). Using \( h_5(S) = c + h_5(S_L) + h_5(S_R) \), and the fact that the (almost) halving line \( \ell \) splits \( S \) such that \( |S_L| = \left\lfloor \frac{n}{2} \right\rfloor \) and \( |S_R| = \left\lfloor \frac{n}{2} \right\rfloor \), we get \( h_5(S) \geq \frac{c_L + c_R}{2} + h_5\left(\left\lfloor \frac{\frac{n}{2}}{2} \right\rfloor\right) + h_5\left(\frac{n-\left\lfloor \frac{n}{2} \right\rfloor}{2}\right) \geq \frac{c_L + c_R}{2} + h_5\left(\frac{n-1}{2}\right) \), and hence, a second lower bound for \( h_5(S) \):

\[
h_5(S) \geq \frac{c_L + c_R}{2} + 2 \cdot h_5\left(\frac{n-1}{2}\right).
\]

Combining this with the bound (1), we obtain

\[
h_5(S) \geq \max \left\{ \left( \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \right), \left( \frac{c_L + c_R}{2} + 2 \cdot h_5\left(\frac{n-1}{2}\right) \right) \right\}. \tag{3}
\]

Note that the first term in inequality (3) is strictly monotonically decreasing in \( \frac{c_L + c_R}{2} \), while the second term is strictly monotonically increasing in \( \frac{c_L + c_R}{2} \).

Thus, the minimum of the lower bound in (3) is reached if both bounds are equal.

\[
\frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} = \frac{c_L + c_R}{2} + 2 \cdot h_5\left(\frac{n-1}{2}\right)
\]

\[
\frac{c_L + c_R}{2} + 2 \cdot h_5\left(\frac{n-1}{2}\right) = \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5\left(\frac{n-1}{2}\right)
\]

Plugging this result for \( \frac{c_L + c_R}{2} \) into the lower bound (2) for \( h_5(S) \), we obtain a lower bound for \( h_5(S) \) for any \( S \) with \( n \) points. Therefore, this also leads to a lower bound for \( h_5(n) \).

\[
h_5(n) \geq \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5\left(\frac{n-1}{2}\right) + 2 \cdot h_5\left(\frac{n-1}{2}\right) = \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5\left(\frac{n-1}{2}\right) \tag{4}
\]

We show by induction that this recursion resolves to \( h_5(n) \geq \frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8} \), for \( n \geq 12 \). We know that \( h_5(12), \ldots, h_5(16) \geq 3 \) and \( h_5(17), \ldots, h_5(23) \geq 4 \) (see first paragraph of this proof). As \( \frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8} \) is monotonically increasing for \( 12 \leq n \leq 23 \), it is sufficient to check the induction base for \( n = 16 \) and \( n = 23 \):

\[
h_5(16) \geq 3 \geq 2.578 \geq \frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8}
\]

\[
h_5(23) \geq 4 \geq 3.609 \geq \frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8}
\]

For \( n \geq 24 \) we insert the claim into the recursive formula:

\[
h_5(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5\left(\frac{n-1}{2}\right) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot \left(\frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8}\right) = \frac{3n}{4} - n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} + \frac{15}{8}
\]

The last inequality is true because \( (n-1)^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} < n^{\left\lfloor \frac{\log_3 n}{3} \right\rfloor + 15} \).

This proves the claim and the theorem as we have:

\[
h_5(n) \geq \frac{3n}{4} - n^{0.87447} + 1.875 = \frac{3n}{4} - o(n).
\]

\[\square\]
3 Empty triangles and convex 4-holes

For this section we are going to use some definitions and notation used in [15, 9, 10]. Let $S$ be a set of $n$ points in the plane in general position. We need to define a total order on the points of $S$. In addition, this order has to define a line $\ell_q$ through every point $q \in S$, such that each point $r \in S$ is either in the closed halfplane “below” $\ell_q$, i.e., $q \leq r$, or in the open halfplane “above” $\ell_q$, i.e., $q < r$. In [10] the points of $S$ are sorted in increasing order of the ordinate $y$. Hence, although the empty triangles and convex 4-holes that this implies that no two points have equal ordinate, observe that, of course any direction is a valid order for the points of $S$. Furthermore, observe that also a cyclic order around some point $p \in ((\partial CH(S)) \cap S)$ is a valid order for the points of $S \setminus \{p\}$, as there exists a line $\ell$ through $p$, such that all points of $S \setminus \{p\}$ are in an open halfplane bounded by $\ell$. This will be crucial for the proof of Lemma 6 where we will order the points of a set $S \setminus \{p\}$ around such a point $p$. Note that, because of the general position assumption for $S$, no two points in $S \setminus \{p\}$ are equivalent in this order. Anyhow, for simplicity, and apart from the aforementioned exception, we will use the order along the ordinate of $S$, as in [10].

Let $P$ be a convex 5-hole spanned by points of $S$ and let $v$ be the top vertex of $P$, i.e., the vertex of $P$ with highest priority. We name an empty triangle generated by $P$ if it is spanned by $v$ and the two vertices of $P$ that are not adjacent (on the boundary of $P$) to $v$. Let $h_{3/5}(S)$ be the number of such triangles determined by $S$, and let $h_{3/5}(n) = \min_{|S|=n} h_{3/5}(S)$ be the number of empty triangles generated by convex 5-holes that every set of $n$ points spans at least. Likewise, we name a convex 4-hole generated by $P$ if it is spanned by all vertices of $P$ except for one of the two vertices of $P$ that are adjacent (on the boundary of $P$) to $v$. Observe that each convex 5-hole generates two convex 4-holes by this definition. Let $h_{4/5}(S)$ be the number of such 4-holes determined by $S$, and let $h_{4/5}(n) = \min_{|S|=n} h_{4/5}(S)$ be the number of convex 4-holes generated by convex 5-holes that every set of $n$ points spans at least.

García [10] recently proved that $h_{3/5}(S) = n^2 - 5n + H + 4 + h_{3/5}(S) \geq n^2 - 5n + H + 4 + h_{3/5}(n)$ and $h_{4/5}(S) = \frac{n^2}{2} - \frac{3n}{2} + H + 3 + h_{4/5}(S) \geq \frac{n^2}{2} - \frac{3n}{2} + H + 3 + h_{4/5}(n)$, where $H$ is the number of points of $(\partial CH(S)) \cap S$. Consequently, this gives $h_{3/5}(n) \geq n^2 - 5n + H + 3 + h_{3/5}(n)$ and $h_{4/5}(n) \geq \frac{n^2}{2} - \frac{3n}{2} + 6 + h_{4/5}(n)$, as $H \geq 3$. Observe that this implies that $h_{3/5}(S)$ and $h_{4/5}(S)$ (and of course $h_{3/5}(n)$ and $h_{4/5}(n)$) do not depend on the chosen order of the points. As changing the order does not change the point set, $h_{3/5}(S)$ and $h_{4/5}(S)$ are of course independent of the order. Furthermore, García proved that the number of empty triangles (or convex 4-holes) not generated by convex 5-holes is an invariant of the point set. Hence, although the empty triangles and convex 4-holes generated by convex 5-holes may change with different orders, their numbers stay the same.

Proving $h_{3/5}(n) \geq 3 \cdot \left\lceil \frac{n^2}{8} \right\rceil$ and $h_{4/5}(n) \geq 6 \cdot \left\lceil \frac{n^2}{8} \right\rceil$, García presented the improved bounds $h_{3/5}(n) \geq n^2 - \frac{3n^2}{2} + \frac{21}{8}$ and $h_{4/5}(n) \geq \frac{n^2}{2} - \frac{3n}{4} - \frac{3}{2}$. We will improve these bounds on $h_{3/5}(n)$ and $h_{4/5}(n)$. Showing that for each convex 5-hole counted in Lemma 1 we may count one empty triangle generated by convex 5-holes and two convex 4-holes generated by convex 5-holes will already give an improved bound for both, $h_{3/5}(n)$ and $h_{4/5}(n)$. But using a slightly adapted version of the proof from Theorem 3 will improve the bound on $h_{4/5}(n)$ even further. To this end we have to first prove the base case, i.e., sets of 10, 11, and 12 points.

Having a close look at the example shown in Figure 3, one can see that as soon as the triangle $\triangle$ (or the convex 4-hole $\bigcirc$) is generated by more than one convex 5-hole, there must exist at least one convex 6-hole. We state this fact in more detail and prove it in the following lemma. Note that a similar approach and figure has been used in [10].

**Lemma 4** Let $S$ be a set of $n \geq 6$ points in the plane in general position. Let $\triangle$ (or $\bigcirc$) be an empty triangle (a convex 4-hole) of $S$. If $\triangle$ (or $\bigcirc$) is generated by at least two convex 5-holes, $\triangle_1$ and $\triangle_2$, of $S$, then there exists at least one convex 6-hole, $\triangle_1$, of $S$, containing $\triangle_1$, and one convex 6-hole, $\triangle_2$, of $S$, containing $\triangle_2$, where $\triangle_1 \cap \triangle_2$ is possible.

**Proof.** See Figure 3 (top). Assume that there exists at least one empty triangle, $\triangle = \langle p_i, p_j, p_k \rangle$, with $p_k$ being the top vertex, that is generated by two different convex 5-holes. Let one of them, $\triangle_1$, be spanned by the points $p_i, p_j, p_{l'}, p_k$, (the points shown as full dots in the figure). As $\triangle$ is generated by another convex 5-hole, $\triangle_2$, there must be at least one additional point in one of the regions $L_h, L_i, R_h, R_i$. Otherwise, the new pentagon would not be empty, not be convex, or $\triangle$ would not be generated by it (recall that $p_k$ must be the highest point). W.l.o.g. assume that there exists at least one point $p_{new}$ in $R_i$. It is easy to see that in this case there exists a convex 4-hole spanned by the points $p_i, p_k, p_R, p_{l'}$ ($p_{l'} = p_{new}$ is possible, but not necessary). Together with $p_j$ and $p_{l'}$ this forms a convex 6-hole which contains $\triangle_1$. Starting the argument with $\triangle$ being generated by $\triangle_2$, proves that also $\triangle_2$ is contained in a convex 6-hole.

The argumentation is analogous for a convex 4-hole, $\bigcirc$, that is generated by two different convex 5-holes. See Figure 3 (bottom). The only difference to the previous case (with $\triangle$) is that the additional point $p_{new}$ can not exist in either $L_i$ or $R_i$, depending on which convex 4-hole (either $\bigcirc = \langle p_i, p_j, p_{L}, p_k \rangle$ or $\bigcirc = \langle p_i, p_j, p_k, p_{R} \rangle$) is considered. The former situation is depicted in Figure 3 (bottom).
Using Lemma 4 we are able to provide the base cases $10 \leq n \leq 12$ for $h_{35}(n)$ and $h_{45}(n)$. The proof is omitted in this extended abstract.

**Lemma 5** Every set of 10, 11, or 12 points in the plane in general position contains (i) at least 1, 2, and 3, respectively, different empty triangles generated by convex 5-holes (i.e., $h_{35}(10) = 1$, $h_{35}(11) = 2$, and $h_{35}(12) = 3$) and (ii) at least 2, 4, and 6, respectively, different convex 4-holes generated by convex 5-holes (i.e., $h_{45}(10) = 2$, $h_{45}(11) = 4$, and $h_{45}(12) = 6$).

These base cases allow a lemma similar to Lemma 1. The proof follows the lines of the proof of Lemma 1 and is omitted in this extended abstract.

**Lemma 6** Every set $S$ of $n$ points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geq 0$ and $t \in \{1, 2, 3\}$) contains at least $h_{35}(n) \geq \left\lceil \frac{3n-27+4t}{7} \right\rceil$ empty triangles generated by convex 5-holes and at least $h_{45}(n) \geq \frac{3n-27+4t}{7}$ convex 4-holes generated by convex 5-holes.

As mentioned above, this lemma already improves the bounds for $h_{35}(n)$ and $h_{45}(n)$. We will further improve the bound for $h_{45}(n)$ in Theorem 8. In the following theorem we state only the bound for $h_{35}(n)$.

**Theorem 7** Every set $S$ of $n \geq 12$ points in the plane in general position contains at least $h_{35}(n) \geq 3 \cdot \lceil \frac{n-12}{7} \rceil + 3 + f(|S_{rem}|) \geq \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles generated by convex 5-holes. The point set $S_{rem} \subseteq S$ is the remainder set with $0 \leq |S_{rem}| \equiv (n - 12) \mod 7 \leq 6$, and $f(0, \ldots, 4) = 0$, $f(5) = 1$, and $f(6) = 2$.

**Proof.** The first inequality in the bound, $h_{35}(n) \geq 3 \cdot \lceil \frac{n-12}{7} \rceil + 3 + f(|S_{rem}|)$, is simply a reformulation of the bound in Lemma 6. The second inequality results from taking the minimum of the first inequality over all possible values for $|S_{rem}|$. (This minimum is obtained by $|S_{rem}| = 4$.)

The basic principles of the proof of the following theorem are the same as in the proof of Theorem 3. The main difference is that, for excluding over-counting, a slightly different counting is needed. The proof is omitted in this extended abstract and we only state the result.

**Theorem 8** Every set $S$ of $n \geq 12$ points in the plane in general position contains at least $h_{45}(n) \geq \frac{5n - 383}{283} + \frac{27}{283}$ convex 4-holes generated by convex 5-holes.

**Remark:** To use the principles of the proof of Theorem 3 also for empty triangles generated by convex 5-holes, a very disadvantageous splitting is necessary to avoid over-counting. This would lead to a bound inferior to the one from Theorem 7.

Recall that García [10] recently proved $h_3(S) \geq n^2 - 5n + H + 4 + h_{35}(n)$ and $h_4(S) \geq \frac{n^2}{2} - \frac{7n}{4} + H + 3 + h_{45}(n)$. Combining these results with Theorem 7 and Theorem 8 we can state the following corollary.

**Corollary 9** Every set $S$ of $n \geq 12$ points in the plane in general position and with $H$ points on the boundary of its convex hull contains at least $h_3(S) \geq n^2 - 5n + H + 4 + \left\lceil \frac{3n-27}{7} \right\rceil$ empty triangles and at least $h_4(S) \geq \frac{n^2}{2} - \frac{9n}{4} + \frac{383}{283} + \frac{19}{283}$ convex 4-holes. Consequently, $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{27}{7}$ and $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641n^{0.926} + \frac{499}{44}$.

### 4 Conclusion

In this paper we improved the lower bounds on the least number $h_k(n)$ of convex $k$-holes any set of $n$ points contains, for $3 \leq k \leq 5$. The question whether there exists a super-linear lower bound for the number of convex 5-holes remains unsettled, though.

Still, we are able to answer two questions that Dehnhardt [6] asked in 1987. Already in [3] a set of 12 points containing only three convex 5-holes has been presented, implying $h_5(12) = 3$. This disproved Dehnhardt’s conjecture of $h_5(12) = 4$. Recall that we know from García [10], that $h_3(S) = n^2 - 5n + H + 4 + h_{35}(S)$ and $h_4(S) = \frac{n^2}{2} - \frac{7n}{4} + H + 3 + h_{45}(S)$, where $h_3(S)$ ($h_4(S)$) is the number of empty triangles (convex 4-holes) generated by convex 5-holes in $S$.

Consider the set $S_{12}$ with $n = 12$ points and $H = 3$, depicted in Figure 4. It can be easily checked that...
this point set contains only the 3 shown convex 5-holes. Hence, \( h_{3|5}(S_{12}) = 3 \) and \( h_{4|5}(S_{12}) = 6 \), as by Lemma 5 \( h_{3|5}(12) = 3 \) and \( h_{4|5}(12) = 6 \). Inserting into the above equations, we get \( h_3(S_{12}) = h_4(S_{12}) = 144 - 60 + 3 + 4 + 3 = 94 \) and \( h_4(S_{12}) = h_5(12) = 72 - 42 + 3 + 6 = 42 \), as \( h_3(12) \geq 94 \) and \( h_4(12) \geq 42 \) (by [6]). Of course, \( h_3(S_{12}) \) and \( h_4(S_{12}) \) can also be derived by counting all empty triangles and convex 4-holes in \( S_{12} \). This disproves two conjectures of Dehnhardt in [6], namely \( h_3(12) = 95 \) and \( h_4(12) = 44 \).

Furthermore, his question for a set of \( n \) points that minimizes at least one of \( h_3(n) \), \( h_4(n) \), and \( h_5(n) \), but not all of them is answered by the set of 12 points presented in [3], which has only 3 convex 5-holes but contains (non-minimal) 95 empty triangles and (non-minimal) 43 convex 4-holes.

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References


