Common Grounds for Modeling Mathematics in Educational Software

Introduction to the Special Track “Convergence on Math Assistants”

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CADGME at Hagenberg Jul.11 09
Outline

1. Variety of Mathematics Assistants (MAs)
   - MAs and Doing Mathematics
   - Example: Bending Lines
   - MAs and In/Formal Mathematics

2. Common Grounds for MAs?
   - “Step” as a “Most General Unifier”?
   - Formalized (= Coded !) Contexts
   - Human Part in Doing Mathematics
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Example: Bending Lines

From a textbook for Technical High Schools (HTL)

Determine the bending line of a beam of length $L$, which consists of homogenous material, which is clamped on one side and which is under constant line load $q_0$.

Hint: Use the constraints $y(0) = 0$, $y'(0) = 0$, $V(0) = q_0 \cdot L$, $M_b(L) = 0$. 

Abb. 7.59
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- **Model:** specify formally
- **Operate:** do steps and justify
- **Interpret:** relate (recur ?) to (1)
- **Communicate:** present, discuss, argument, reason

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A “Step” in doing math

A *step* starts from a *Context* and produces a result which can be *justified* . . .

\[
\text{step} : \text{Context} \times \text{State} \times \text{Interact} \rightarrow \text{Context} \times \text{State} \times \text{Result}
\]

. . . where *State* concerns technicalities of MAs and

*Interaction*: compound operation
- draw a geometric object (e.g. ortho-center of a triangle)
- call a CAS command (e.g. \( \text{Integrate } x^3 + x^2 + x + 1 \, dx \))
- . . .

atomic operation
- substitute a value for a variable
- apply a rule (e.g. \( \int 2x \, dx = x^2 + c \)) to transform a formula
- . . .

Formality of *Context* constrains rigor of justification!
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**Formal specification**

**Specification** of the problem on the bending line:

- **in**: length \( L \), function \( q_0 \)
- **pre**: \( L > 0 \land q_0 \text{ is integrable in } x \)
- **out**: function \( y(x) \)
- **post**: \( y(0) = 0 \land y'(0) = 0 \land V(0) = q_0 \cdot L \land M_b(L) = 0 \)

where \( V \) and \( M_b \) are constant function symbols in the theory of “bending lines”.

**Formal Specification** required for *mechanical* steps!
Formal specification

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The human part in formulas

(i) Problem solving creates a **Result**:

\[
\text{solve} : \text{Theory} \times \text{Context} \times \text{Specification} \rightarrow \text{Context} \times \text{Result}
\]

where

\[
\text{Specification} = \text{Input} \times \text{Precondition} \times \text{OutputVar} \times \text{Postcondition}
\]

and \( \text{post}(\text{in}, \text{res}) \) holds for \( \text{pre}(\text{in}) \)

(ii) Theorem proving constructs a **Theorem**:

\[
\text{prove} : \text{Theory} \times \text{Context} \times \text{Predicate} \rightarrow \text{Theory} \times \text{Theorem}
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(i) expands knowledge **outside** the formal model - “applied mat”
(ii) expands knowledge **within** the formal domain – “pure math”
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“Common Grounds”?

Some particular answers . . .:

1. Convergence on concepts for learning with MAs?
   - Step is a basic notion, less or more formal!
   - . . .

2. Convergence on technology of MAs?
   - Serve MAs with Logic-based math-engines!
   - . . .

3. Convergence on principles of e-learning?
   - We need a formal domain model of e-learning!
   - . . .

. . . looking forward to many other answers in the track!
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